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Spinor Methods in Conformal Killing Transport

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Abstract

The equations of conformal Killing transport are discussed using tensor and spinor methods. It is shown that, in Minkowski space-time, the equations for a null conformal Killing vector ξ^a are completely determined by the corresponding spinor ω^A and its covariant derivative, which defines a spinor $\pi_{A'}$. In conformally flat space-time, the covariant derivative of $\pi_{A'}$ is also involved. Some applications to twistor theory are briefly mentioned.

1. *Conformal Transformations and Conformal Rescalings*

Let M be a differentiable manifold with metric tensor g. A conformal transformation from M into itself is a C^∞ map f where $f: M \to M$, and $f^* \mathrm{g} = \Omega^2 \mathrm{g}$, Ω being a smooth, real-valued, positive scalar function on M; if $\Omega = 1$, then f is an isometry. In the case of Minkowski space-time, all transformations of coordinates which transform the Minkowski metric* η_{ab} into a metric g_{ab}, where $g_{ab} = \Omega^2 \eta_{ab}$, form a 15-parameter group—the conformal group of Minkowski space-time; this group contains the Poincaré and Lorentz groups as subgroups. In fact, the conformal group mentioned here more correctly gives the symmetries of a compactified Minkowski space-time, consisting of Minkowski space-time itself together with a null cone at infinity (Penrose, 1967), since under the inversion $ds^2 \to d\hat{s}^2 = \Omega^2\, ds^2$, where $\Omega^2 = 1/(x^a x_a)^2$, the null cone at the origin is sent to infinity (x^a is the position vector of a point x with respect to the origin). All transformations of the conformal group can be obtained from space and time reflections together with the transformations of the restricted conformal group—this last being a 15-parameter Lie group, generated by the infinitesimal conformal transform-

* The abstract index notation is used; lower case Roman indices denote tensors, upper case Roman indices (primed or unprimed) denote spinors, and Greek indices denote twistors; components in some basis are denoted by bold face indices.

ations $x^a \to x^a + \epsilon\xi^a$, where ϵ is an infinitesimal parameter and ξ^a is a conformal Killing vector which satisfies the conformal Killing equation:

$$\pounds_\xi g_{ab} = \phi g_{ab} \tag{1.1}$$

Here ϕ is a scalar function on the manifold; the condition for an isometry now is that $\phi = 0$, in which case the equation is called Killing's equation and $\boldsymbol{\xi}$ is a Killing vector; if ϕ is a constant, a homothetic transformation is defined. The set $CK(M)$ of all conformal Killing vectors on a manifold M forms a Lie algebra over $\mathbb{R}$, and the set of all Killing vectors $K(M)$ on M is a subalgebra of $CK(M)$. The following result is well known: The group of conformal transformations of an n-dimensional Riemannian manifold is a Lie transformation group of dimension at most $\frac{1}{2}(n + 1)(n + 2)$, provided that $n \geqslant 3$ (Kobayashi & Nomizu, 1963).

A conformal rescaling of the metric tensor g_{ab} on a space-time M is a transformation of g_{ab}, thus

$$g_{ab} \to \hat{g}_{ab} = \Omega^2 g_{ab}$$

also

$$g^{ab} \to \hat{g}^{ab} = \Omega^{-2} g^{ab} \tag{1.2}$$

These transformations form an infinite-dimensional Abelian group, which preserves angles between vectors and the null-cone structure of the space-time.

A tensor $A_{b\cdots d}^{a\cdots c}$ is said to be conformally invariant if it is invariant under (1.2), i.e.,

$$A_{b\cdots d}^{a\cdots c} \to \hat{A}_{b\cdots d}^{a\cdots c} = A_{b\cdots d}^{a\cdots c}$$

A tensor $B_{b\cdots d}^{a\cdots c}$ is said to be a conformal density of weight N if, under (1.2),

$$B_{b\cdots d}^{a\cdots c} \to \hat{B}_{b\cdots d}^{a\cdots c} = \Omega^N B_{b\cdots d}^{a\cdots c}$$

(the metric itself is thus a conformal density). A flat space-time theory which is conformally invariant and also Poincaré invariant is then also invariant under the 15-parameter conformal group, since Poincaré transformations of Minkowski space-time become conformal transformations according to a conformally rescaled flat metric.

2. *Conformal Killing Transport*

Let M be a connected pseudo-Riemannian manifold with metric g_{ab}; the conformal Killing equation may then be written*:

$$\nabla_{(a}\xi_{b)} = \phi g_{ab} \tag{2.1}$$

where

$$\phi = (1/n)\nabla_a\xi^a, \qquad n = \dim M$$

* Round brackets denote symmetry, square brackets denote skew symmetry.

Also, by commutation of derivatives and using the properties of the curvature tensor and the Bianchi identities, it follows that

$$\begin{aligned} \nabla_a \xi_b &= F_{ab} + \phi g_{ab}; \qquad F_{ab} = F_{[ab]} \\ \nabla_a \phi &= K_a \\ \nabla_a F_{bc} &= R_{bcad}\xi^d - 2g_{a[b}K_{c]} \\ \nabla_a K_b &= \xi^d \nabla_d P_{ab} + 2\phi P_{ab} + 2P_{d(a}F_{b)}{}^d \end{aligned} \tag{2.2}$$

In these equations $P_{ab} = (1/2)\,R_{ab} - (1/12)\,Rg_{ab}$. Hence for any curve y through a point $x \in M$, with tangent vector η^a,

$$\begin{aligned} \eta^a \nabla_a \xi_b &= \eta^a\{F_{ab} + \phi g_{ab}\} \\ \eta^a \nabla_a \phi &= \eta^a\{K_a\} \\ \eta^a \nabla_a F_{bc} &= \eta^a\{R_{bcad}\xi^d - 2g_{a[b}K_{c]}\} \\ \eta^a \nabla_a K_b &= \eta^a\{\xi^d \nabla_d P_{ab} + 2\phi P_{ab} + 2P_{d(a}F_{b)}{}^d\} \end{aligned} \tag{2.3}$$

These equations are called the conformal Killing transport equations (Geroch, 1969); they define the quadruple $(\xi^a, \phi, F_{ab}, K_a)$ along the curve y through x, the quadruple being given at x. Let V denote the $\frac{1}{2}(n+1)(n+2)$-dimensional space of all quadruples $(\xi^a, \phi, F_{ab}, K_a)$ at x; a conformal holonomy group can be defined at x—this being the group of all linear transformations on V obtained by conformal Killing transport of the quadruple $(\xi^a, \phi, F_{ab}, K_a)$ along each closed curve y starting at x—the quadruples which are mapped into themselves under this group of transformations correspond to the conformal Killing vector fields on M.

A set of Killing transport equations can be obtained from (2.3) by putting $\phi = 0$ in these equations; the result is

$$\begin{aligned} \eta^a \nabla_a \xi_b &= \eta^a\{F_{ab}\} \\ \eta^a \nabla_a F_{bc} &= \eta^a\{R_{bcad}\xi^d\} \end{aligned} \tag{2.4}$$

and in this case ξ^a is a Killing vector.

In what follows it will be convenient to introduce the Weyl conformal tensor C_{abcd} and to define the vector $p_a = K_a - P_{ad}\xi^d$. The equations (2.3) then become

$$\begin{aligned} \eta^a\{\nabla_a \xi_b\} &= \eta^a\{F_{ab} + \phi g_{ab}\} \\ \eta^a\{\nabla_a \phi\} &= \eta^a\{p_a + P_{ad}\xi^d\} \\ \eta^a\{\nabla_a F_{bc}\} &= \eta^a\{C_{bcad}\xi^d + 2P_{a[b}\xi_{c]} - 2g_{a[b}p_{c]} \\ \eta^a\{\nabla_a p_b\} &= \eta^a\{\xi^d \nabla_{[d}P_{a]b} + \phi P_{ab} + P_{da}F_b{}^d\} \end{aligned} \tag{2.5}$$

and this last can be rewritten:

$$\eta^a\{\nabla_a p_b\} = \eta^a\{\xi^d \nabla^x C_{xbad} + \phi P_{ab} + P_{da}F_b{}^a\}$$

since $\nabla_{[d}P_{a]b} = \nabla^x C_{xbad}$.

3. *Solutions in Minkowski Space-time*

In Minkowski space-time $n = 4$, and $R_{abcd} = 0$. The conformal Killing transport equations are then much simplified and can be written

$$\begin{aligned}
\nabla_a \xi_b &= F_{ab} + \phi g_{ab} \\
\nabla_a \phi &= p_a \\
\nabla_a F_{bc} &= -2g_{a[b}p_{c]} \\
\nabla_a p_b &= 0
\end{aligned} \tag{3.1}$$

Also $p_a = K_a$ since $P_{ab} = 0$. Introducing a Minkowski coordinate system $\{x^a\}$, the equations can be successively integrated, beginning with the last, as follows:

$$\begin{aligned}
p_{1a} &= p_{0a} \\
F_{1ab} &= F_{0ab} - 2x_{[a}p_{0b]} \\
\phi_1 &= \phi_0 + x^a p_{0a} \\
\xi_1{}^a &= \xi_0{}^a + F_{0b}{}^a x^b + \phi_0 x^a + x^a(x^b p_{0b}) \\
&\qquad -\tfrac{1}{2}P_0{}^a(x^b x_b)
\end{aligned} \tag{3.2}$$

where x^a is the position vector of x_1 with respect to the origin 0, and $(\xi_0{}^a, \phi_0, F_{0ab}, p_{0a})$ and $(\xi_1{}^a, \phi_1, F_{1ab}, p_{1a})$ are the corresponding values of the quadruple $(\xi^a, \phi, F_{ab}, p_a)$ at the points in question. In the set of equations (3.2), $\xi_0{}^a$ defines the translations (four parameters) and F_{0ab} defines the Lorentz transformations (six parameters); also ϕ_0 defines the dilations (one parameter) and p_{0a} defines the so-called "uniform acceleration" transformations (four parameters—see Penrose & MacCallum, 1972 for comment on these). Hence the conformal symmetry group of Minkowski space-time is a 15-parameter group—the conformal group previously referred to in Sec. 1; $\xi_0{}^a$ and F_{0ab} together define the 10-parameter Poincaré group, which is the metric-preserving subgroup of the conformal group.

4. *Spinor Methods—Minkowski Space-Time*

Let now ξ^a be a null conformal Killing vector; then it can be written in spinor terms thus: $\xi^a = \omega^A \bar{\omega}^{A'}$. Then

$$\nabla_a \xi_b = \nabla_{AA'}(\omega_B \bar{\omega}_{B'}) = \phi \epsilon_{AB}\epsilon_{A'B'} + F_{ABA'B'} \tag{4.1}$$

translating the first of equations (3.1) into spinors. But $F_{ab} = F_{[ab]}$; therefore there exists a symmetric spinor $\mu_{AB} = \mu_{(AB)}$ such that

$$F_{ab} = F_{ABA'B'} = \epsilon_{A'B'}\mu_{AB} + \epsilon_{AB}\bar{\mu}_{A'B'} \tag{4.2}$$

also, from (3.1), it follows that

$$\nabla_{(A'}{}^{(A}\omega^{B)}\bar{\omega}_{B')} = 0$$

which implies the existence of a constant spinor $\pi_{A'}$ (Penrose, 1967) such that:

$$\nabla_{AA'}\omega_B = -i\epsilon_{AB}\pi_{A'} \tag{4.3}$$

with complex conjugate

$$\nabla_{AA'}\bar{\omega}_{B'} = i\epsilon_{A'B'}\bar{\pi}_A \tag{4.4}$$

and

$$\nabla_{AA'}\pi_{B'} = 0 \tag{4.5}$$

It follows that

$$\mu_{AB} = i\omega_{(A}\bar{\pi}_{B)} \tag{4.6}$$

and

$$\phi = (i/2), (\omega^C\bar{\pi}_C - \bar{\omega}^{C'}\pi_{C'}) \tag{4.7}$$

Further, using (4.3), (4.4), (4.5),

$$\nabla_{AA'}\phi = \bar{\pi}_A\pi_{A'} \tag{4.8}$$

and hence $p_a = \bar{\pi}_A\pi_{A'}$ is a null vector.

Collecting these results together, for a null conformal Killing vector in Minkowski space-time, the following relationships hold:

$$\begin{aligned}
\xi^a &= \omega^A\bar{\omega}^{A'} \\
\phi &= (i/2)(\omega^C\bar{\pi}_C - \bar{\omega}^{C'}\pi_{C'}), \qquad \text{where } \pi_{A'} = (i/2)\nabla_{BA'}\omega^B \\
F_{Ab} &= \epsilon_{A'B'}\mu_{AB} + \epsilon_{AB}\bar{\mu}_{A'B'}, \qquad \text{where } \mu_{AB} = i\omega_{(A}\bar{\pi}_{B)} \\
p_a &= \bar{\pi}_A\pi_{A'}
\end{aligned} \tag{4.9}$$

The spinor equivalent equations of (3.1) can then be written thus:

$$\begin{aligned}
\nabla_{AA'}\omega_B &= -i\epsilon_{AB}\pi_{A'} \\
\nabla_{AA'}\phi &= \bar{\pi}_A\pi_{A'} \\
\nabla_{AA'}\mu_{BC} &= \epsilon_{A(B}\bar{\pi}_{C)}\pi_{A'} \\
\nabla_{AA'}\pi_{B'} &= 0
\end{aligned} \tag{4.10}$$

A straightforward calculation shows also that the equations (3.2) can be written in terms of ω^A, ϕ, μ_{AB}, $\pi_{A'}$ as follows:

$$\begin{aligned} \omega_1{}^A &= \omega_0{}^A - ix^{AB'}\pi_{0B'} \\ \phi_1 &= \phi_0 + \bar{\pi}_{0A}\pi_{0A'}x^{AA'} \\ \mu_{1AB} &= \mu_{0AB} + x_{CA}{}^{C'}\bar{\pi}_{0B)}\pi_{C'} \\ \pi_{1A'} &= \pi_{0A'} \end{aligned} \tag{4.11}$$

It is evident that in spinor terms the conformal Killing transport equations in Minkowski space-time for a null conformal Killing vector are completely determined from a knowledge of the corresponding spinor ω^A and its covariant derivative, which gives the spinor $\pi_{A'}$, since the quantities ϕ, μ_{AB} (hence F_{ab}), and p_a can be defined in terms of these; for this reason ω^A is called a conformal Killing spinor. The vector p_a and the tensor F_{ab} have the same form as the momentum and angular momentum tensors, respectively, of a system of particles in Minkowski space-time (Penrose & MacCallum, 1972).

5. *Conformal Rescalings*

Under a conformal rescaling of the metric η_{ab}, i.e., $\eta_{ab} \to \hat{\eta}_{ab} = g_{ab}$, where $g_{ab} = \Omega^2\eta_{ab}$, the conformal Killing vector ξ^a is invariant: $\xi^a \to \hat{\xi}^a = \xi^a$. Also, the conformal Killing transport equations are covariant, so that

$$\begin{aligned} F_{ab} \to \hat{F}_{ab} &= \Omega^2(F_{ab} + 2\gamma_{[a}\xi_{b]}) \\ \phi \to \hat{\phi} &= \phi + \gamma_c\xi^c \\ p_a \to \hat{p}_a &= p_a + \gamma_c F_a{}^c + \gamma_a\phi \\ &\quad + \xi^b\gamma_a\gamma_b - \tfrac{1}{4}\gamma_c\gamma^c\xi_a \end{aligned}$$

where

$$\gamma_a = \Omega^{-1}\nabla_a\Omega$$

Since η_{ab} is a flat metric, the metric g_{ab} is conformally flat, i.e., the Weyl tensor $C_{abcd} = 0$, but the curvature tensor R_{abcd} is nonzero and can in fact be expressed in terms of the metric tensor and the tensor P_{ab}. The conformal Killing transport equations are therefore [from (2.5)]

$$\begin{aligned} \eta^a\{\nabla_a\xi_b\} &= \eta^a\{F_{ab} + \phi g_{ab}\} \\ \eta^a\{\nabla_a\phi\} &= \eta^a\{p_a + P_{ad}\xi^d\} \\ \eta^a\{\nabla_a F_{bc}\} &= \eta^a\{2P_{a[b}\xi_{c]} - 2g_{a[b}p_{c]}\} \\ \eta^a\{\nabla_a p_b\} &= \eta^a\{\phi P_{ab} + P_{da}F_b{}^d\} \end{aligned} \tag{5.1}$$

These equations can also be expressed in terms of the spinors introduced earlier, thus if $\xi^a = \omega^A \bar{\omega}^{A'}$, the equation (4.3) is still valid, but now $\pi_{A'}$ is not a constant spinor, and in fact

$$\nabla_{AA'}\pi_{B'} = iP_{ABA'B'}\omega^B \tag{5.2}$$

The equations (5.1) then become, in terms of ω^A and $\pi_{A'}$,

$$\begin{aligned}
\eta^{AA'}\{\nabla_{AA'}\omega_B\} &= \eta^{AA'}\{-i\epsilon_{AB}\pi_{A'}\} \\
\eta^{AA'}\{\nabla_{AA'}\phi\} &= \eta^{AA'}\{\bar{\pi}_A\pi_{A'} + P_{ABA'B'}\omega^B\bar{\omega}^{B'}\} \\
\eta^{AA'}\{\nabla_{AA'}\mu_{BC}\} &= \eta^{AA'}\{\pi_{A'}\epsilon_{A(B}\bar{\pi}_{C)} + i\omega_{(B}P_{|A|C)A'D'}\bar{\omega}^{D'}\}^* \\
\eta^{AA'}\{\nabla_{AA'}\pi_{B'}\} &= \eta^{AA'}\{iP_{ABA'B'}\omega^B\}
\end{aligned} \tag{5.3}$$

ξ^a, ϕ, F_{ab}, p_a have the same spinor form as before, and it therefore follows that if ξ^a is null, in a conformally flat space-time, p_a is null.

The equations (4.3) and (5.2) are conformally covariant with

$$\hat{\omega}^A = \omega^A$$

$$\hat{\pi}_{A'} = \pi_{A'} + i\gamma_{BA'}\omega^B$$

and

$$\hat{\mu}_A{}^B = \mu_A{}^B + \tfrac{1}{2}\gamma_{C'}{}^B\omega_A\bar{\omega}^{C'} + \tfrac{1}{2}\gamma_{AC'}\omega^B\bar{\omega}^{C'} \tag{5.4}$$

The spinor equations (5.3) cannot be maintained in a space-time which is not conformally flat, since then there is a consistency condition on ω^A

$$\nabla^{X'(C}\nabla_{X'}{}^B\omega^{A)} = -\,\omega^X\Psi_X{}^{ABC}$$

where Ψ_{ABCD} is the Weyl spinor. The equations (5.3) are consistent if and only if $\omega^X\Psi_{XABC} = 0$.

6. *Applications in Twistor Theory*

The equations (4.3), (4.5), and (5.2) are of interest in twistor theory—the equation $\nabla^{A'(A}\omega^{B)} = 0$, which holds in Minkowski space-time, is called the twistor equation: the spinor field ω^A then completely defines a twistor Z^α, since the twistor equation has the general solution

$$\omega_1{}^A = \omega_0{}^A - ix^{AA'}\pi_{A'}$$

from the first equation of (4.11), and the twistor Z^α is represented by the spinor pair $(\omega^A, \pi_{A'})$ (Penrose & MacCallum, 1972).

In local twistor theory, a twistor covariant derivative $\nabla_\rho{}^\sigma$ is introduced; the spinor parts of this derivative are of the form of the equations (4.3) and (5.2). It can easily be shown then that the local twistor covariant derivative is integrable in Minkowski space-time and that the equation $\nabla_\rho{}^\sigma Z^\alpha = 0$ is

* The notation here is that the index A, being contained thus: $|A|$, is to be excluded from the symmetry operation.

conformally invariant (Dighton, 1974). The spinor equations (4.3) and (5.2) can thus be used to define the operation of local twistor transport, which has the same tensor form as the operation of conformal Killing transport in a conformally flat space-time. This will be enlarged upon in a later paper.

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